

The Jacobian of a nonorientable Klein surface, II

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Abstract. The aim here is to continue the investigation in [1] of Jacobians of a Klein surface and also to correct an error in [1].

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1. Introduction

This note is a continuation of the study of the Jacobian of nonorientable, compact Klein surfaces that we started in [1]. To explain our results we need to recall the set-up in [1]. Let Y be a nonorientable, connected compact Klein surface (the changes of coordinates in Y are either holomorphic or anti-holomorphic), and let X be the canonical double cover of Y given by local orientations. It is well-known that X is a Riemann surface. The non-trivial deck transformation, which we will denote by σ , for covering $X \rightarrow Y$ is an anti-holomorphic involution on X with Y as the quotient.

The Jacobian of X was considered in [1] from three different points of view: (1) as the dual of the space of holomorphic 1-forms, (2) as the divisor class group, and (3) as the group of isomorphism classes of degree zero line bundles. In each case, the involution σ induces an involution in the corresponding description of the Jacobian. In Theorem 4.2 of [1] it was stated that the Jacobian of Y can be identified with the set of line bundles L such that $\sigma^* \bar{L}$ is holomorphically isomorphic to L . However, the proof of that result is not correct. It was asserted in the proof that the isomorphism of L given by $\sigma^* \bar{\alpha} \circ \alpha$ can be so chosen that it is an involution of the total space of L . In this note we show that the isomorphism can be chosen to be of order four, and there may not be any isomorphism of order two.

Let \mathcal{G} denote the group of line bundles L over X such that $\sigma^* \bar{L}$ is holomorphically isomorphic to L . Let $\mathcal{G}_0 \subset \mathcal{G}$ be the subgroup defined by all L such that the isomorphism of L with $\sigma^* \bar{L}$ can be chosen to be an involution of the total space of L .

The correct version of Theorem 4.2 of [1] is as follows:

Theorem 2.5. *The image of the homomorphism ρ in p. 147, eq. (3) of [1] coincides with $J^0(X) \cap \mathcal{G}_0$. If the genus of X is even, then the image of the homomorphism ρ (p. 147, eq. (3) of [1]) coincides with $J^0(X) \cap \mathcal{G}$. If the genus of X is odd, then the image of ρ is a subgroup of $J^0(X) \cap \mathcal{G}$ of index two.*

This theorem is proved in § 1. In §§ 2 and 3 we considered two other approaches to the Jacobian mentioned above (dual space to the space of holomorphic forms and divisors)

and proved this result in those settings. Thus we get three different (but clearly equivalent) proofs of the theorem, which shows that one can consider any of these three ways of describing the Jacobian of X to study the Jacobian of Y .

2. Real points of the Jacobian and real algebraic line bundles

Let X be a compact connected Riemann surface equipped with a fixed-point free anti-holomorphic involution σ . So $\sigma: X \rightarrow X$ is a diffeomorphism with the property that if $J(X) \in C^\infty(X, \text{End}(T^{\mathbb{R}}X))$ is the almost complex structure on X , then $\sigma^*J(X) = -J(X)$. Note that this implies that σ is orientation reversing.

Let L be a holomorphic line bundle over X such that $\sigma^*\bar{L}$ is holomorphically isomorphic to L . We note that $\sigma^*\bar{L}$ is a holomorphic line bundle (see § 4 of [1] for the details).

Take an isomorphism

$$\alpha: L \rightarrow \sigma^*\bar{L}. \quad (1)$$

Therefore,

$$(\sigma^*\bar{\alpha}) \circ \alpha: L \rightarrow L \quad (2)$$

is an automorphism of the holomorphic line bundle L . Indeed, we have

$$\sigma^*\bar{\alpha}: \sigma^*\bar{L} \rightarrow \sigma^*\overline{\sigma^*\bar{L}} = L.$$

We note that the holomorphic line bundle $\sigma^*\overline{\sigma^*\bar{L}}$ is canonically identified with L .

Let $c \in \mathbb{C} \setminus \{0\}$ be the nonzero complex number such that

$$(\sigma^*\bar{\alpha}) \circ \alpha = c \cdot \text{Id}_L, \quad (3)$$

where $(\sigma^*\bar{\alpha}) \circ \alpha$ is the automorphism in (2).

We will show that c is actually a real number.

Let

$$M := L \otimes \sigma^*\bar{L}$$

be the holomorphic line bundle over X . We note that

$$\sigma^*\bar{M} = \sigma^*\bar{L} \otimes \sigma^*\overline{\sigma^*\bar{L}} = \sigma^*\bar{L} \otimes L = M.$$

Let

$$\tau: M \rightarrow \sigma^*\bar{M} \quad (4)$$

be the above isomorphism. It is easy to see that $(\sigma^*\bar{\tau}) \circ \tau$ is the identity automorphism of the line bundle M .

Next we observe that the tensor product of two homomorphisms $\alpha \otimes \sigma^*\bar{\alpha}$ is an automorphism of the holomorphic line bundle M . Indeed, α sends the line bundle L to $\sigma^*\bar{L}$ and $\sigma^*\bar{\alpha}$ sends $\sigma^*\bar{L}$ to $\sigma^*\overline{\sigma^*\bar{L}} = L$. Therefore, $\alpha \otimes \sigma^*\bar{\alpha}$ sends the line bundle $L \otimes \sigma^*\bar{L} =: M$ to itself.

Consequently,

$$\alpha \otimes \sigma^*\bar{\alpha} = d \cdot \text{Id}_M, \quad (5)$$

where d is a nonzero complex number.

Let

$$\delta := \tau \circ (\alpha \otimes \sigma^* \bar{\alpha}): M \rightarrow \sigma^* \bar{M} \quad (6)$$

be the isomorphism, where τ is the isomorphism in (4) and $\alpha \otimes \sigma^* \bar{\alpha}$ is the automorphism of M in (5).

We consider the automorphism

$$(\sigma^* \bar{\delta}) \circ \delta: M \rightarrow M,$$

where δ is defined in (6). From (5) it follows immediately that

$$(\sigma^* \bar{\delta}) \circ \delta = |d|^2 \cdot \text{Id}_M.$$

On the other hand, from (3) it follows that

$$(\sigma^* \bar{\delta}) \circ \delta = c^2 \cdot \text{Id}_M.$$

Since $c^2 \cdot \text{Id}_M = (\sigma^* \bar{\delta}) \circ \delta = |d|^2 \cdot \text{Id}_M$, we have $c^2 = |d|^2$. Consequently, $c \in \mathbb{R}$, where c is the scalar in (3).

As $c \in \mathbb{R} \setminus \{0\}$, the isomorphism

$$\alpha_0 := \frac{\alpha}{\sqrt{|c|}}: L \rightarrow \sigma^* \bar{L}$$

has the property that

$$(\sigma^* \bar{\alpha}_0) \circ \alpha_0 = \pm \text{Id}_L,$$

where α is the isomorphism in (1).

Therefore, we have proved the following lemma.

Lemma 2.1. Let L be a holomorphic line bundle over X such that the holomorphic line bundle $\sigma^ \bar{L}$ is isomorphic to L . Then there is an isomorphism*

$$\alpha: L \rightarrow \sigma^* \bar{L}$$

such that $\sigma^ \bar{\alpha} \circ \alpha$ is either Id_L or $-\text{Id}_L$.*

In the proof of Theorem 4.2 of [1] it was erroneously asserted that for any L as in Lemma 2.1, there is always an isomorphism

$$\alpha: L \rightarrow \sigma^* \bar{L}.$$

such that $\sigma^* \bar{\alpha} \circ \alpha$ is Id_L . In Lemma 2.3 we will see that this is not the case.

Remark 2.2. The pair (X, σ) corresponds to a geometrically connected smooth projective curve defined over \mathbb{R} without any real points. A holomorphic line bundle L over X of degree d with $\sigma^* \bar{L}$ holomorphically isomorphic to L corresponds to a real point of the Jacobian $J^d(X)$ of line bundles of degree d . If there is an isomorphism $\alpha: L \rightarrow \sigma^* \bar{L}$ such that $\sigma^* \bar{\alpha} \circ \alpha = \text{Id}_L$, then L corresponds to an algebraic line bundle defined over the real algebraic curve.

Let \mathcal{G} denote the group of all holomorphic line bundles L over X such that $\sigma^*\bar{L}$ is holomorphically isomorphic to L . The group operation is given by tensor product. Let

$$\mathcal{G}_0 \subset \mathcal{G}$$

be the subgroup consisting of all $L \in \mathcal{G}$ admitting an isomorphism

$$\alpha: L \rightarrow \sigma^*\bar{L}$$

such that $(\sigma^*\bar{\alpha}) \circ \alpha = \text{Id}_L$.

Using Lemma 2.1 we have a character

$$\lambda: \mathcal{G} \rightarrow \mathbb{Z}/2\mathbb{Z} \quad (7)$$

defined by the following condition: any $L \in \mathcal{G}$ admits an isomorphism

$$\alpha: L \rightarrow \sigma^*\bar{L}$$

such that $(\sigma^*\bar{\alpha}) \circ \alpha = \lambda(L) \cdot \text{Id}_L$ after identifying $\mathbb{Z}/2\mathbb{Z}$ with ± 1 . Note that for any nonzero complex number z and any isomorphism α as above, we have $(\sigma^*\bar{z\alpha}) \circ (z\alpha) = |z|^2 \cdot (\sigma^*\bar{\alpha}) \circ \alpha$. Using this it follows immediately that the above map λ is well-defined.

The kernel of the homomorphism λ in (7) coincides with the subgroup \mathcal{G}_0 .

Lemma 2.3. The homomorphism λ defined in (7) is surjective.

Proof. Since the anti-holomorphic involution σ of X does not have any fixed points, there exists a meromorphic function f on X such that

$$f \cdot \overline{f \circ \sigma} = -1 \quad (8)$$

(see [5] for the construction of f). We note that $\overline{f \circ \sigma}$ is also a holomorphic function over X .

From (8) it follows that f is a nonconstant function. Let D_0 (respectively, D_1) be the effective divisor on X defined by the zeros (respectively, poles) of f . From (8) it follows that

$$\sigma(D_0) = D_1. \quad (9)$$

Let $L = \mathcal{O}_X(D_0)$ be the holomorphic line bundle over X defined by the divisor D_0 . Similarly, let $L' = \mathcal{O}_X(D_1)$ be the holomorphic line bundle defined by D_1 . From (9) it follows that $\sigma^*\bar{L} = L'$.

The meromorphic function f defines an isomorphism of $\mathcal{O}_X(D_0)$ with $\mathcal{O}_X(D_1)$. Let α be the isomorphism of L with $\sigma^*\bar{L} = L'$ given by f . From (8) it follows that $(\sigma^*\bar{\alpha}) \circ \alpha = -\text{Id}_L$. Therefore, the character λ in (7) is nontrivial. This completes the proof of the lemma. \square

The following lemma gives the parity of the degree of any line bundle $L \in \mathcal{G}$ over X with $\lambda(L) = -1$.

Lemma 2.4. Let $L \in \mathcal{G}$ be a holomorphic line bundle over X such that $L \notin \mathcal{G}_0$, i.e., $\lambda(L) = -1$. Then $\text{degree}(L) \equiv \text{genus}(X) + 1 \pmod{2}$.

Proof. Replacing L by the tensor product of L with the holomorphic line bundle $\mathcal{O}_X(D)$, where D is an effective divisor on X such that $\sigma(D) = D$ and $\text{degree}(D)$ is sufficiently large, we may assume that $H^1(X, L) = 0$. Note that since σ is fixed-point free, the degree of D is even.

Fix an isomorphism $\alpha: L \rightarrow \sigma^* \bar{L}$ such that $\sigma^* \bar{\alpha} \circ \alpha = -\text{Id}_L$. This isomorphism α induces a conjugate linear automorphism $\hat{\alpha}$ of the complex vector space $H^0(X, L)$ such that $\hat{\alpha}^2 = -\text{Id}$. From this it follows immediately that the complex dimension of $H^0(X, L)$ is even.

The Riemann–Roch theorem says

$$\dim H^0(X, L) - \dim H^1(X, L) = \text{degree}(L) - \text{genus}(X) + 1.$$

Since $\dim H^0(X, L)$ is even and $H^1(X, L) = 0$, the lemma follows from the Riemann–Roch theorem. \square

Remark 2.5. Take any line bundle $L \in \mathcal{G} \setminus \mathcal{G}_0$. From Lemma 2.4 it follows that $\text{degree}(L) = 2d_0$ if the genus of X is odd, and $\text{degree}(L) = 2d_0 + 1$ if the genus is even. Take a divisor $D_0 = \sum_{i=1}^{d_0} x_i X$ of degree d_0 such that $\sigma(D_0) = D_0$ and $\{x_1, \dots, x_{d_0}\}$ are distinct points. Consider the line bundle $L' := L \otimes \mathcal{O}_X(-D_0)$ over X . Note that $L' \in \mathcal{G} \setminus \mathcal{G}_0$, and $\text{degree}(L')$ is zero or one depending on the parity of $\text{genus}(X)$.

Consider the intersection $J^0(X) \cap \mathcal{G}_0$. The homomorphism ρ in p. 147, eq. (3) of [1] maps to it. Combining Lemmas 2.3 and 2.4 we have the following corrected version of Theorem 4.2 of [1].

Theorem 2.6. *The image of the homomorphism ρ in p. 147, eq. (3) of [1] coincides with $J^0(X) \cap \mathcal{G}_0$. If the genus of X is even, then the image of the homomorphism ρ in p. 147, eq. (3) of [1] coincides with $J^0(X) \cap \mathcal{G}$. If the genus of X is odd, then the image of ρ is a subgroup of $J^0(X) \cap \mathcal{G}$ of index two.*

3. The Jacobian as dual space of holomorphic forms

Following the setting and notation of [1], let $J_1(X)$ denote the Jacobian of X obtained as the quotient of $H^0(X, \Omega)^*$ (the dual of the space of holomorphic forms on X) by the action of $H_1(X, \mathbb{Z})$. In Proposition 3.1 of [1] we showed that if $\{\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g\}$ is a canonical (symplectic) basis of $H_1(X, \mathbb{Z})$ satisfying

$$\sigma_{\#}(\gamma_j) = \gamma_j, \tag{10}$$

then the associated basis of holomorphic form, $\{\omega_1, \dots, \omega_g\}$ ($\int_{\gamma_j} \omega_k = \delta_{jk}$) is invariant, that is, $\overline{\sigma^*(\omega_j)} = \omega_j$. Here $\sigma_{\#}$ and σ^* denote the maps induced by σ in homology and 1-forms respectively.

Let $\pi: \mathbb{C}^g \rightarrow J_1(X) \cong \mathbb{C}^g / \mathbb{Z}^g$ be the natural projection, and let σ_1 denote the involution induced by σ on $J_1(X)$. In [1] we showed that the lift of σ_1 to \mathbb{C}^g , with the above basis, is given by conjugation. The fixed points of σ_1 are therefore given by the solutions of the equation

$$\bar{z} = z + n + Pm, \tag{11}$$

where P is the period matrix and n and m are points in \mathbb{Z}^g . We obtain the solutions to this equation in two different ways, depending on whether the genus of X is even or odd.

Let us first fix some notation. Let I_n denote the $n \times n$ identity matrix, and K_n the $n \times n$ matrix with entries equal to 1 in the anti-diagonal and 0 in all other entries.

Assume first that X has even genus. The fundamental group of the Klein surface Y has one relation given by $c^2(a_1 b_1 a_1^{-1} b_1^{-1}) \cdots (a_{g-1} b_{g-1} a_{g-1}^{-1} b_{g-1}^{-1}) = e$, from which we get, by a simple topological argument (see [2]), a symplectic basis of X , say \mathcal{B} . It is easy to see that the action of $\sigma_\#$ on $H_1(X, \mathbb{Z})$ with respect to \mathcal{B} is given by the matrix K_{2g} . We make a change of basis using the $2g \times 2g$ matrix

$$C = \begin{pmatrix} -I_g & (I + K)_g \\ -K_g & K_g \end{pmatrix};$$

let $\mathcal{B}_1 = \mathcal{B}C = \{\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g\}$ be the new basis. Since C satisfies $C^t J C = J$, where $J = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ is the standard intersection matrix, we have that \mathcal{B}_1 is symplectic. The action of $\sigma_\#$ with respect to \mathcal{B}_1 is given by

$$C^{-1} \sigma_\# C = \begin{pmatrix} I_g & (-2I - K)_g \\ 0 & -I_g \end{pmatrix},$$

so \mathcal{B}_1 satisfies condition (10).

Let P denote the period matrix with respect to this new basis of $H_1(X, \mathbb{Z})$ and the associated basis of holomorphic forms; denote by p_{jk} its entries. Let A be the $g \times g$ matrix given by $A = -2I_g - K_g$ and denote by a_{jk} the entries of A . Then we have

$$\overline{p_{kj}} = \int_{\delta_j} \overline{\omega_k} = \int_{\delta_j} \sigma^*(\omega_k) = \int_{\sigma_\#(\delta_j)} \omega_k = a_{jk} - p_{jk}.$$

So the real part of P is equal to $\operatorname{Re} P = -I - \frac{1}{2}K$.

Since the imaginary part of P is invertible (see for example Proposition III.2.8 of [3]), we have that any point z of \mathbb{C}^g can be written as $z = x + Py$, where x and y are points in \mathbb{R}^g . Considering the real and imaginary parts of (11) we obtain the following two equations:

$$\begin{cases} 0 & = n + (\operatorname{Re} P)m, \\ -(\operatorname{Im} P)y & = (\operatorname{Im} P)y + (\operatorname{Im} P)m. \end{cases}$$

From the first equation, using the expression of $\operatorname{Re} P$ obtained above and the fact that all entries of n are integers we get that the entries of m are even integers. Since $\operatorname{Im} P$ is invertible, the second equation gives $y = -\frac{1}{2}m$, which implies that y has integer entries. The set of fixed points of σ_1 is therefore given by the projection of

$$\mathcal{T} = \{x + Py; x \in \mathbb{R}^g, y \in \mathbb{Z}^g\} \subset \mathbb{C}^g$$

to $J_1(X)$.

The odd genus case is handled in a similar way (although the computations are a little more complicated): using topological arguments we get a symplectic basis \mathcal{B} of $H_1(X, \mathbb{Z})$; the action of $\sigma_\#$ with respect to this basis is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{g-1} \\ 0 & 0 & -1 & 0 \\ 0 & K_{g-1} & 0 & 0 \end{pmatrix}.$$

We change basis in $H_1(X, \mathbb{Z})$ to $\mathcal{B}_1 = \mathcal{B}C$, where

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -I_{g-1} & 0 & (I+K)_{g-1} \\ 0 & 0 & 1 & 0 \\ 0 & -K_{g-1} & 0 & K_{g-1} \end{pmatrix}.$$

One can easily show that \mathbb{B}_1 is symplectic and satisfies condition (10). The period matrix satisfies the identity $P = A - \overline{P}$, where A is the matrix $A = \begin{pmatrix} 0 & 0 \\ 0 & (-2I-K)_{g-1} \end{pmatrix}$. Splitting (11) in its real and imaginary parts as above we obtain the following pairs of equations (we used again the fact that $\text{Im} P$ is invertible):

$$\begin{cases} n &= -(\text{Re } P)m, \\ y &= -\frac{1}{2}m. \end{cases}$$

From the first equation we get that if $m' = (m_1, \dots, m_g) \in \mathbb{Z}^g$ then m_2, \dots, m_g are even integers. So the set of fixed points of σ_1 has two components, given by the projections of the following two sets to $J_1(X)$:

$$\mathcal{T}_1 = \{x + Py; x \in \mathbb{R}^g, y \in \mathbb{Z}^g\} \subset \mathbb{C}^g,$$

$$\mathcal{T}_2 = T_1 + P \left(\frac{1}{2}, 0, \dots, 0 \right)' = \mathcal{T}_1 + \frac{1}{2}p_1 \subset \mathbb{C}^g.$$

Here p_1 is the first period. We refer the reader to [4] where similar results are stated, although with different computations. The Jacobian of Y can then be identified with the subgroup $\pi(\mathcal{T}_1)$ of $J_1(X)$.

4. Divisors

Let $J_0(X)$ be the Jacobian of X given as the divisor class group, that is degree zero divisors quotiented by the principal divisors. We have a natural involution in $J_0(X)$ defined by $\sigma_0([D]) = [\sigma^*(D)]$, where square brackets denote equivalence classes and σ^* the natural extension of σ to divisors. It was shown in [1] that the involution σ_0 is equivalent to σ_1 by the Abel–Jacobi map. A divisor class $[D]$ is fixed by σ_0 if D is linearly equivalent to $\sigma^*(D)$; that is, there exists a holomorphic function $f: X \rightarrow \widehat{\mathbb{C}}$ such that $D - \sigma^*(D) = \text{div}(f)$. This implies that $\text{div}(\overline{f \circ \sigma}) = -\text{div}(f)$, so there exists a constant c such that $f \cdot \overline{f \circ \sigma} = c$ (here \cdot denotes multiplication of complex numbers). But then $\overline{f \circ \sigma} \cdot f = \overline{c} = c$, so c is a real number. Multiplying f by a number we can assume that $c = \pm 1$.

If $c = 1$, let h be the function $h = f + 1$. Then $\overline{h \circ \sigma} = h/f$ so that $f = h/(\overline{h \circ \sigma})$, which implies $\text{div}(f) = \text{div}(h) - \text{div}(\overline{h \circ \sigma}) = \text{div}(h) - \sigma^*(\text{div}(h))$. Let E be the divisor $E = D - \text{div}(h)$. Then we have that E is linearly equivalent to D and

$$\sigma^*(E) = \sigma^*(D) - \sigma^*(\text{div}(h)) = (D - \text{div}(f)) - (\text{div}(h) - \text{div}(f)) = E.$$

Therefore D is linearly equivalent to a divisor (E) that comes from the surface Y .

If $f \cdot \overline{f \circ \sigma} = -1$ it can be easily seen that D is not linearly equivalent to any divisor that comes from Y . So we get that the set of fixed points on σ_0 consists of the following disjoint sets:

$$\mathcal{T}_1 = \{[D]; \deg(D) = 0, \sigma^*(D) = D\} \text{ and}$$

$$\mathcal{T}_2 = \{[D]; \deg(D) = 0, D - \sigma^*(D) = \operatorname{div}(f), f \cdot \overline{f \circ \sigma} = 1\}.$$

Clearly \mathcal{T}_1 is not empty.

If X has even degree, then by Lemma 2.3 we have that \mathcal{T}_2 is empty. In the case of odd degree, let f be as in (8) and let D_0 and D_1 denote the divisors defined by the zeroes and poles of f , respectively. Equation (8) gives $\sigma^*(D_1) = D_0$. Again by Lemma 2.3 we have that D_0 has even degree. Let E be a divisor with $\deg(E) = \deg(D_0)$ and $E = \sigma^*(E)$ (observe that this last condition, since σ does not have fixed points, forces the degree of E to be even). Define $\mathcal{X} = E - D_0$. Then \mathcal{X} is a degree zero divisor satisfying $\mathcal{X} - \sigma^*(\mathcal{X}) = \operatorname{div}(f)$, so $[\mathcal{X}] \in \mathcal{T}_2$. This shows that \mathcal{T}_2 is not empty in the case of X having odd degree. It is easy to see that $[D] \in \mathcal{T}_1$ if and only if $[D + \mathcal{X}] \in \mathcal{T}_2$, so \mathcal{T}_2 is the translation of \mathcal{T}_1 by \mathcal{X} .

We can identify \mathcal{T}_1 with the Jacobian of Y (and with the component \mathcal{T}_1 obtained in the previous version, via the Abel–Jacobi mapping).

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